EXACT SOLUTION OF PROBLEM OF DEFLECTION OF AN ANISOTROPIC PLATE WITH AN APERTURE

T. L. Martynovich

UDC 539.3

The exact solution of the problem of the deflection of an anisotropic plate weakened by an aperture is known only for the case in which the aperture has the shape of a circle or an ellipse [1, 2]. An exact solution has not been derived for any other aperture shapes. Approximate methods [3-6] which are widespread for the case of multiply connected anisotropic plates [7] are applied to the determination of the bending moments in an anisotropic plate near an aperture differing little from an elliptical or circular one.

§1. Let a thin anisotropic plate of thickness h having at each point a plane of elastic symmetry parallel to the median plane xOy occupy an infinite volume S with an aperture bounded by a simple smooth closed contour L which is described by the equation $x + iy = R\left(e^{i\theta} - \sum_{k=1}^{N} c_k e^{-ik\theta}\right)$. Let us discuss the first fundamental problem in which bending moments m(s) are applied to the edge of the aperture L of the plate, and let the bending moments and torques at parts of the plate distant from the aperture be bounded: $M_x^\infty = M_1$, $M_y^\infty = M_2$, and $H_{xy}^\infty = M_{12}$.

On the basis of the formulas of the deflection theory for anisotropic plates [1, 2] we write the boundary conditions in the differential form

$$dV = -m(s)dt \quad (t \in L), \tag{1.1}$$

with

$$V = \sum_{j=1}^{2}\left[\left(q_j + i\frac{p_j}{\mu_j}\right)\varphi_j(z_j) + \left(\bar{q}_j + i\frac{\bar{p}_j}{\bar{\mu}_j}\right)\overline{\varphi_j(z_j)}\right], \tag{1.2}$$

$$\lim_{|z_j|\to\infty} \varphi_j'(z_j) = A^{(j)} \quad (j = 1, 2),$$

where $\varphi_j(z_j)$ are analytic functions describing the stress state in the plate, $z_j = x + \mu_j y$ ($j = 1, 2$) are generalized complex variables which vary in the regions S_j obtainable from the region S by the appropriate affine transformations; $\mu_j = \alpha_j + i\beta_j$ are the roots of the characteristic equation; p_j, q_j are known constant quantities [1, 2]; t is the affix of a point of the contour L; and $A^{(j)}$ are constants which are expressed in terms of the bending moments and torques in the plate at infinity.

The contours of the apertures of the regions S_j of the variables $z_j = x + \mu_j y$ are denoted by L_j, and the affixes of their points are denoted by t_j ($j = 1, 2$). The affixes of the points of the contours L_j and the contour L are related by the affine relation

$$t_j = \frac{1 - i\mu_j}{2}t + \frac{1 + i\mu_j}{2}\bar{t} \quad (j = 1, 2). \tag{1.3}$$

Let us convert the boundary conditions (1.1) to integral form [8]

$$\begin{aligned} \int_L F(t)\,dV &= -\int_L F(t)\,m(t)\,dt, \\ \int_L \overline{F(t)}dV &= -\int_L \overline{F(t)}m(t)\,dt, \end{aligned} \tag{1.4}$$

where F(t) is the limiting value of the arbitrary function F(z) of the variable $z = x + iy$, which is holomorphic in the region S of the plate.

L'vov. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 168-177, September-October, 1977. Original article submitted September 6, 1976.

 0021-8944/77/1805-0728$07.50

Let the regulat function which performs a conformal mapping of the exterior of the unit circle γ ($|\zeta| > 1$) onto the interior of the contour L of the region S be of the form

$$z = \omega(\zeta) = R\left(\zeta + \sum_{k=1}^{N} c_k \zeta^{-k}\right)(\omega'(\zeta) \neq 0, |\zeta| \geqslant 1), \tag{1.5}$$

with [9]

$$\sum_{k=1}^{N} k|c_k|^2 < 1.$$

Varying the constants R, c_k, and N in Eq. (1.5), one can obtain an aperture in the form of a circle, an ellipse, oval, curved triangle, quadrangle, and other figures.

Equations (1.3) will take the form

$$t_j = \frac{R_j}{R}\left[\omega(\sigma) + m_j \overline{\omega(\sigma)}\right] \quad (t_j \in L_j, \sigma \in \gamma) \tag{1.6}$$

when the mapping function (1.5) is taken into account, where

$$R_j = \frac{R(1 - i\mu_j)}{2}; \quad m_j = \frac{1 + i\mu_j}{1 - i\mu_j} \quad (j = 1, 2).$$

Equations (1.6) are the limiting values of the functions

$$z_j = \omega_j(\zeta_j) = \frac{R_j}{R}\left[\omega(\zeta_j) + m_j \bar{\omega}\left(\frac{1}{\zeta_j}\right)\right](z_j \in S_j, |\zeta_j| \geqslant 1),$$

which are regular in the regions $|\zeta_j| \leq 1$ except the points $\zeta_j = \infty$, where they have a pole of order N. The functions $\omega_j(\zeta_j)$ and $\omega_j'(\zeta_j)$ have zeros located outside the unit circle $\gamma(|\zeta_j| \leq 1)$ whose number is equal to $N - 1$ [9]. Outside γ only $\omega_j(\zeta_j) \neq 0$ and $\omega_j'(\zeta_j) \neq 0$ for $N = 0$ (circular aperture) and $N = 1$ ($|c_1| < 1$) (elliptical aperture).

At large $|z_j|$ the functions $\varphi_j(z_j)$ are of the form [1, 2]

$$\varphi_j(z_j) = D^{(j)} \ln z_j + A^{(j)} z_j + A_0'^{(j)} + O\left(\frac{1}{z_j}\right) \quad (j = 1, 2).$$

The constants $D^{(j)}$ are expressed in terms of the components of the principal moment of the external forces applied to the boundary L according to known formulas (the principal vector $P_z = 0$) [2].

Introducing the notation $\varphi_j[\omega_j(\zeta_j)] = \varphi_{*j}(\zeta_j)$, we find

$$\varphi_j'(z_j) = \frac{\varphi_{*j}'(\zeta_j)}{\omega_j'(\zeta_j)} \quad (j = 1, 2), \tag{1.7}$$

where

$$\omega_j'(\zeta_j) = \frac{R_j}{R}\left[\omega'(\zeta_j) - \frac{m_j}{\zeta_j^2} \bar{\omega}'\left(\frac{1}{\zeta_j}\right)\right]. \tag{1.8}$$

The functions $_{*j}(\zeta_j)$ are bounded in the regions $1 \leq |\zeta_j| < \infty$, and they have a pole of order N at the points $\zeta_j = \infty$. The last assertions follow from the conditions (1.2) imposed on the functions $\varphi_j(z_j)$ at infinity

$$\lim_{|z_j| \to \infty} \varphi_j'(z_j) = \lim_{|\zeta_j| \to \infty} \frac{\varphi_{*j}'(\zeta_j)}{\omega_j'(\zeta_j)} = A^{(j)} \quad (j = 1, 2), \tag{1.9}$$

and from the boundedness of the expression

$$2\frac{\partial W}{\partial z} = \sum_{j=1}^{2}\left[(1 + i\mu_j)\varphi_{*j}(\zeta_j) + (1 + i\bar{\mu}_j)\overline{\varphi_{*j}(\zeta_j)}\right] \quad (1 \leqslant |\zeta_j| < \infty),$$

where W is the bending deflection of the plate.

Consequently, the functions $_{*j}(\zeta_j)$, which are bounded in the region $1 \leq |\zeta_j| < \infty$, can be represented at sufficiently large $|\zeta_j|$ in the form of series (the unbounded terms are discarded)

$$\varphi_{*j}(\zeta_j) = D^{(j)} \ln \zeta_j + \sum_{k=1}^{N} a_k^{(j)} \zeta_j^k + \sum_{k=0}^{\infty} A_k^{(j)} \zeta_j^{-k} \quad (j = 1, 2). \tag{1.10}$$

The single-valued functions (1.7) do not have other singular points besides the poles coinciding with the zeros of the function $\omega_j'(\zeta_j)$. Consequently, they are meromorphic functions of the variables ζ_j. In the case under discussion (1.10) are rational-fractional functions by virtue of the representations (1.5).

With the appropriate definition of the functions $\varphi_{*j}'(\zeta_j)$ one can ensure that the functions (1.7) will be bounded outside the unit circle γ. To accomplish this it is sufficient to require that the zeros of the function $\varphi_{*j}'(\zeta_j)$ coincide outside γ with the zeros of the function $\omega_j'(\zeta_j)$.

Thus the functions $\varphi_{*j}'(\zeta_j)$ should satisfy the conditions

$$\varphi_{*j}'(\zeta_j^{(n)}) = 0 \quad (n = 1, 2, \ldots, N-1), \quad (j = 1, 2), \tag{1.11}$$

where $\zeta_j^{(n)}$ are the roots of the equations

$$\omega_j'(\zeta_j) = 0 \quad (j = 1, 2)$$

larger than unity in absolute value ($|\zeta_j^{(n)}| > 1$).

The functions (1.7) take the form

$$\varphi_j'(z_j) = \frac{D^{(j)} + \sum\limits_{k=1}^{N} k a_k^{(j)} \zeta_j^k - \sum\limits_{k=0}^{\infty} k A^{(j)} \zeta_j^{-k}}{R_j\left[\left(\zeta_j - \sum\limits_{k=1}^{N} k c_k \zeta_j^{-k}\right) - \frac{m_j \bar{R}}{R}\left(\zeta_j^{-1} - \sum\limits_{k=1}^{N} k \bar{c}_k \zeta_j^k\right)\right]} \tag{1.12}$$

on the basis of Eqs. (1.5), (1.8), and (1.10).

The conditions (1.9) and (1.11) with the expansions (1.10) taken into account are written in the form

$$\sum_{k=0}^{N} k A_k^{(j)} (\zeta_j^{(n)})^{-k} - \sum_{k=1}^{N-1} k a_k^{(j)} (\zeta_j^{(n)})^k = N a_N^{(j)} (\zeta_j^{(n)})^N + D^{(j)} -$$
$$- \sum_{k=N+1}^{\infty} k A_k^{(j)} (\zeta_j^{(n)})^{-k} \qquad (n = 1, 2, \ldots, N-1), \quad (j = 1, 2), \tag{1.13}$$

with

$$a_N^{(j)} = R_j m_j \bar{c}_N \bar{R} R^{-1} A^{(j)} \ (N > 1).$$

Here $\zeta_j^{(n)}$ are the roots of the equations

$$\zeta_j - \sum_{k=1}^{N} k c_k \zeta_j^{-k} - \frac{m_j \bar{R}}{R}\left(\zeta_j^{-1} - \sum_{k=1}^{N} k \bar{c}_k \zeta_j^k\right) = 0 \quad (j = 1, 2) \tag{1.14}$$

larger than unity in absolute value ($|\zeta_j^{(n)}| > 1$).

In the transformed region the boundary conditions (1.4) will take the form

$$\begin{aligned} \int_\gamma F_*(\sigma)\, dV &= -\int_\gamma F_*(\sigma)\, m(\sigma)\, \omega'(\sigma)\, d\sigma, \\ \int_\gamma \overline{F_*(\sigma)} dV &= -\int_\gamma \overline{F_*(\sigma)}\, m(\sigma)\, \omega'(\sigma)\, d\sigma, \end{aligned} \tag{1.15}$$

where $F_*(\zeta) = F[\omega(\zeta)]$ is an arbitrary function holomorphic outside γ.

The limiting value of the function V on γ is, according to Eqs. (1.2) and (1.10), equal to

$$V = \sum_{j=1}^{2} \sum_{k=1}^{N} \left[\left(q_j + i\frac{p_j}{\mu_j}\right) a_k^{(j)} \sigma^k + \left(\bar{q}_j + i\frac{\bar{p}_j}{\bar{\mu}_j}\right) \bar{a}_k^{(j)} \sigma^{-k}\right] +$$
$$+ \sum_{j=1}^{2} \sum_{k=0}^{\infty} \left[\left(q_j + i\frac{p_j}{\mu_j}\right) A_k^{(j)} \sigma^{-k} + \left(\bar{q}_j + i\frac{\bar{p}_j}{\bar{\mu}_j}\right) \bar{A}_k^{(j)} \sigma^k\right] + D^* \ln \sigma, \tag{1.16}$$

with

$$D^* = -\frac{1}{2\pi i} \int_{\gamma+} m(\sigma)\, \omega'(\sigma)\, d\sigma = -\frac{M_x^* + i M_y^*}{2\pi i}.$$

Let us represent the arbitrary function $F_*(\zeta)$ in the form of a series

$$F_*(\zeta)=\sum_{n=0}^{\infty}E_n\zeta^{-n}. \tag{1.17}$$

Let us introduce Eqs. (1.16) and (1.17) into the boundary conditions (1.15) and perform an integration along the closed contour of γ. At the same time setting all E_j except E_n equal to zero, we obtain a system of linear algebraic equations in the coefficients of the expansion of the desired functions (1.10) of the form

$$\sum_{j=1}^{2}\left[\left(q_j+i\frac{p_j}{\mu_j}\right)A_n^{(j)}+\left(\bar{q}_j+i\frac{\bar{p}_j}{\bar{\mu}_j}\right)\bar{a}_n^{(j)}\right]=f_n,$$
$$\sum_{j=1}^{2}\left[\left(\bar{q}_j+i\frac{\bar{p}_j}{\bar{\mu}_j}\right)\bar{A}_n^{(j)}+\left(q_j+i\frac{p_j}{\mu_j}\right)a_n^{(j)}\right]=g_n\quad(n=1,2,\ldots,\infty), \tag{1.18}$$

with $a_n^{(j)}=0$ for $n>N$ and

$$f_n=\frac{1}{2\pi in}\int_{\gamma+}\sigma^n m(\sigma)\,\omega'(\sigma)\,d\sigma,\quad g_n=-\frac{1}{2\pi in}\int_{\gamma+}\sigma^{-n}m(\sigma)\,\omega'(\sigma)\,d\sigma.$$

From the system (1.18) we find for $n>N$

$$A_n^{(1)}=\frac{q_2-i\dfrac{p_2}{\mu_2}}{2i\left(q_2\dfrac{p_1}{\mu_1}-q_1\dfrac{p_2}{\mu_2}\right)}f_n-\frac{q_2+i\dfrac{p_2}{\mu_2}}{2i\left(q_2\dfrac{p_1}{\mu_1}-q_1\dfrac{p_2}{\mu_2}\right)}\bar{g}_n,$$

$$A_n^{(2)}=\frac{q_1+i\dfrac{p_1}{\mu_1}}{2i\left(q_2\dfrac{p_1}{\mu_1}-q_1\dfrac{p_2}{\mu_2}\right)}\bar{g}_n-\frac{q_1-i\dfrac{p_1}{\mu_1}}{2i\left(q_2\dfrac{p_1}{\mu_1}-q_1\dfrac{p_2}{\mu_2}\right)}f_n\quad(n>N).$$

Having added Eqs. (1.13) to the system (1.18) ($n = 1, 2, \ldots, N$), we obtain the final system of linear algebraic equations of order $4N-2$ [N is the largest negative power in the expansion of the mapping function (1.5)] for the determination of the remaining coefficients of the expansion of the functions (1.10).

In the case of the second fundamental problem in which the values of the bending deflection W and the normal derivatives $\partial W/\partial n$ of points of the contour L of the region S are specified and the bending moments and torques in the plate at infinity are bounded, the system of Eqs. (1.18) is replaced by the following one:

$$\sum_{j=1}^{2}\left[(1+i\mu_j)A_n^{(j)}+(1+i\bar{\mu}_j)\bar{a}_n^{(j)}\right]=f_n^*,$$
$$\sum_{j=1}^{2}\left[(1+i\bar{\mu}_j)\bar{A}_n^{(j)}+(1+i\mu_j)a_n^{(j)}\right]=g_n^*\qquad(n=1,2,\ldots,\infty), \tag{1.19}$$

with $a_n^{(j)}=0$ for $n>N$ and

$$f_n^*=-\frac{2}{2\pi in}\int_{\gamma+}\sigma^n d\left[\frac{\partial W}{\partial t}\right],\quad g_n^*=\frac{2}{2\pi in}\int_{\gamma+}\sigma^{-n}d\left[\frac{\partial W}{\partial t}\right].$$

If the principal moment of the external forces causing the specified deflection W and the inclination angle of the curved surface $\partial W/\partial n$ at points of the contour L are equal to zero, then the constants $D^{(j)}=0$. Having solved the system (1.19) for $n>N$, we find

$$A_n^{(1)}=\frac{1-i\mu_2}{2i(\mu_1-\mu_2)}f_n^*-\frac{1+i\mu_2}{2i(\mu_1-\mu_2)}\bar{g}_n^*,$$
$$A_n^{(2)}=\frac{1+i\mu_1}{2i(\mu_1-\mu_2)}\bar{g}_n^*-\frac{1-i\mu_1}{2i(\mu_1-\mu_2)}f_n^*\qquad(n>N).$$

Having assigned Eqs. (1.13) to the system (1.19) ($n = 1, 2, \ldots, N$), we obtain the final system of linear algebraic equations of order $4N-2$ for the determination of the remaining coefficients of the expansion of the functions (1.10).

In the case in which an absolutely rigid core is soldered into the aperture L of the plate, $W=2\,\mathrm{Re}\,(\varepsilon_0 t)+W_0(t\in L)$, where ε_0 is a complex quantity, and, consequently, $f_n^*=0$ and $g_n^*=0$.

Everything that has been said above can be applied with obvious insignificant changes to the case of a finite region S mapped onto the circle $|\zeta|\le 1$ by a function of the form

$$z=\omega(\zeta)=R\left(\zeta+\sum_{k=2}^{N}c_k\zeta^k\right).$$

§2. Let us discuss the deflection of an orthotropic plate with a triangular aperture. Let us direct the x and y coordinate axes parallel to the principal elasticity directions. At infinity the plate is deflected by the moments $M_x^\infty = M_1$, $M_y^\infty = M_2$, and $H_{xy}^\infty = 0$. The edge of the aperture L of the plate is not loaded ($m = 0$).

In this case $N = 2$, $c_1 = 0$, $c_2 = \bar{c}_2$, $D^{(j)} = 0$, $f_n = 0$, $g_n = 0$, $R = \bar{R}$, $\mu_2 = -\bar{\mu}_1$, $m_2 = \bar{m}_1$, $R_2 = \bar{R}_1$, $p_2 = \bar{p}_1$, $q_2 = \bar{q}_1$, $r_2 = -\bar{r}_1$, $A_n^{(2)} = \bar{A}_n^{(1)}$, and $a_n^{(2)} = \bar{a}_n^{(1)}$. The system of algebraic equations (1.13) and (1.18) will be of the sixth order of the following form (the coefficients $A_n^{(j)}$ and $a_n^{(j)}$ are complex quantities):

$$\mathrm{Re}\left[\left(q_1 - i\frac{p_1}{\mu_1}\right)A_n^{(1)} + \left(q_1 + i\frac{p_1}{\mu_1}\right)a_n^{(1)}\right] = 0,$$

$$\mathrm{Re}\left[\left(q_1 + i\frac{p_1}{\mu_1}\right)A_n^{(1)} + \left(q_1 - i\frac{p_1}{\mu_1}\right)a_n^{(1)}\right] = 0,$$

$$2A_2^{(1)} + A_1^{(1)}\zeta_1^{(1)} - a_1^{(1)}\left(\zeta_1^{(1)}\right)^3 = 2a_2^{(1)}\left(\zeta_1^{(1)}\right)^4 \quad (n = 1,2),$$

where

$$a_2^{(1)} = \frac{R_1 m_1 c_2\left(\bar{q}_1 M_x^\infty - \bar{p}_1 M_y^\infty\right)}{2(\bar{p}_1 q_1 - p_1 \bar{q}_1)}; \quad a_2^{(2)} = \bar{a}_2^{(1)}.$$

Here $\zeta_1^{(1)}$ ($\zeta_2^{(1)} = \bar{\zeta}_1^{(1)}$) is the root of Eq. (1.14).

$$2c_2 m_1 \zeta_1^4 + \zeta_1^3 - m_1\zeta_1 - 2c_2 = 0\,, \tag{2.1}$$

larger in absolute value than unity ($|\zeta_1^{(1)}| > 1$).

The stress functions are, according to (1.12), of the form

$$\varphi_j'(z_j) = \frac{2a_2^{(j)}\zeta_j^4 + a_1^{(j)}\zeta_j^3 - A_1^{(j)}\zeta_j - 2A_2^{(j)}}{R_j\left[\left(\zeta_j^3 - 2c_2\right) + m_j\left(2c_2\zeta_j^4 - \zeta_j\right)\right]} \quad (j = 1, 2).$$

Numerical values of the bending moments M_θ (in fractions of M) are given in Table 1 at some points of the contour of the triangular aperture ($c_2 = 0.25$) of a plywood panel having the following values of the complex parameters $\mu_1 = \alpha + i\beta$ and $\mu_2 = -\bar{\mu}_1$ [1]:

$$\alpha = 1.04,\ \beta = 1.55,\ \nu_1 = 0.31,\ \nu_2 = 0.026, \text{ if } E_x = E_{\max} \text{ and } \alpha = 0.299,\ \beta = 0.444,\ \nu_1 = 0.026,\ \nu_2 = 0.31, \text{ if } E_x = E_{\min}. \tag{2.2}$$

The roots $\zeta_1^{(1)}$ of Eq. (2.1) are larger than unity in absolute value ($|\zeta_1^{(1)}| > 1$) and accordingly equal to ($c_2 = 0.25$)

$$\zeta_1^{(1)} = 3.603 + i2.938,\quad \zeta_2^{(1)} = \bar{\zeta}_1^{(1)} \quad \text{if} \quad E_x = E_{\max} \text{ and } \zeta_1^{(1)} = -3.570 + i3.068,\quad \zeta_2^{(1)} = \bar{\zeta}_1^{(1)} \quad \text{if} \quad E_x = E_{\min}.$$

§3. Let us consider the deflection of an orthotropic plate with a square aperture. Let us take the principal elasticity directions as the direction of the x and y axes. At infinity the plate is deflected by the moments $M_x^\infty = M$, $M_y^\infty = M$, and $H_{xy}^\infty = 0$. The edge of the aperture L of the plate is not loaded ($m = 0$).

In the case under discussion $N = 3$, $c_1 = 0$, $c_2 = 0$, $c_3 = \bar{c}_3$, $R = \bar{R}$, $D^{(j)} = 0$, $f_n = 0$, $g_n = 0$, $\mu_2 = -\bar{\mu}_1$, $m_2 = \bar{m}_1$, $R_2 = \bar{R}_1$, $p_2 = \bar{p}_1$, $q_2 = \bar{q}_1$, $r_2 = -\bar{r}_1$, $A_n^{(2)} = \bar{A}_n^{(1)}$, $a_n^{(2)} = \bar{a}_n^{(1)}$, and are equal to zero with even indices n When c_3 is positive, the vertices of the square lie on the x and y axes, and when c_3 is negative, the sides of the square are parallel to the coordinate axes.

The system of algebraic equations (1.3) and (1.18) will, when the symmetry of the problem is taken into account, be of sixth order of the following form:

$$\mathrm{Re}\left[\left(q_1 - i\frac{p_1}{\mu_1}\right)A_n^{(1)} + \left(q_1 + i\frac{p_1}{\mu_1}\right)a_n^{(1)}\right] = 0,$$

$$\mathrm{Re}\left[\left(q_1 + i\frac{p_1}{\mu_1}\right)A_n^{(1)} + \left(q_1 - i\frac{p_1}{\mu_1}\right)a_n^{(1)}\right] = 0,$$

$$3A_3^{(1)} + A_1^{(1)}\left(\zeta_1^{(k)}\right)^2 - a_1^{(1)}\left(\zeta_1^{(k)}\right)^4 = 3a_3^{(1)}\left(\zeta_1^{(k)}\right)^6$$
$$(n = 1,3),\ (k = 1,2),$$

where

$$a_3^{(1)} = \frac{R_1 m_1 c_3\left(\bar{q}_1 M_x^\infty - \bar{p}_1 M_y^\infty\right)}{2\left(\bar{p}_1 q_1 - p_1\bar{q}_1\right)}; \quad a_3^{(2)} = \bar{a}_3^{(1)}.$$

Here $\zeta_1^{(k)}$ ($\zeta_2^{(k)} = \bar{\zeta}_1^{(k)}$) are roots of Eq. (1.14),

TABLE 1

θ, rad	$M_x^\infty = M$, $M_y^\infty = 0$		$M_x^\infty = 0$, $M_y^\infty = M$		θ, rad	$M_x^\infty = M$, $M_y^\infty = 0$		$M_x^\infty = 0$, $M_y^\infty = M$	
	$E_x = E_{max}$	$E_x = E_{min}$	$E_x = E_{max}$	$E_x = E_{min}$		$E_x = E_{max}$	$E_x = E_{min}$	$E_x = E_{max}$	$E_x = E_{min}$
0	0,231	1,327	2,720	6,715	$19\pi/36$	1,981	1,384	1,372	0,085
$\pi/36$	0,216	0,385	2,410	4,913	$21\pi/36$	3,923	1,839	2,824	0,098
$\pi/18$	0,189	0,061	1,661	1,967	$22\pi/36$	5,142	2,120	2,999	0,124
$\pi/9$	0,210	0,514	0,311	0,531	$23\pi/36$	5,536	2,279	1,016	0,176
$\pi/6$	0,313	0,695	—0,049	0,309	$2\pi/3$	2,862	1,882	—0,269	0,320
$2\pi/9$	0,420	0,775	0,031	0,236	$13\pi/18$	0,416	—1,447	1,433	1,785
$5\pi/18$	0,518	0,827	0,184	0,195	$27\pi/36$	0,289	—0,876	1,475	2,336
$\pi/3$	0,623	0,876	0,326	0,165	$28\pi/36$	0,239	0,316	1,429	2,329
$7\pi/18$	0,764	0,941	0,462	0,138	$31\pi/36$	0,199	1,664	1,279	1,868
$4\pi/9$	1,002	1,047	0,638	0,112	$17\pi/18$	0,194	1,805	1,211	1,635
$\pi/2$	1,499	1,235	0,999	0,091	π	0,194	1,811	1,199	1,595

TABLE 2

$M_x^\infty = M$, $M_y^\infty = 0$, $H_{xy}^\infty = 0$									
θ, rad	$c_3 = -\frac{1}{9}$		$c_3 = \frac{1}{9}$		θ, rad	$c_3 = -\frac{1}{9}$		$c_3 = \frac{1}{9}$	
	$E_x = E_{max}$	$E_x = E_{min}$	$E_x = E_{max}$	$E_x = E_{min}$		$E_x = E_{max}$	$E_x = E_{min}$	$E_x = E_{max}$	$E_x = E_{min}$
0	0,133	1,482	0,260	3,218	$11\pi/36$	2,192	1,596	0,466	0,596
$\pi/36$	0,132	1,477	0,251	2,347	$\pi/3$	2,341	1,593	0,551	0,643
$\pi/18$	0,132	1,455	0,231	1,056	$13\pi/36$	2,178	1,531	0,690	0,739
$\pi/9$	0,131	1,224	0,225	0,527	$7\pi/18$	1,986	1,466	0,955	0,914
$\pi/6$	0,146	0,004	0,269	0,544	$4\pi/9$	1,726	1,377	2,865	1,553
$2\pi/9$	0,270	—0,933	0,330	0,561	$17\pi/36$	1,664	1,356	5,001	1,882
$5\pi/18$	1,366	1,372	0,409	0,575	$\pi/2$	1,644	1,349	5,944	2,019

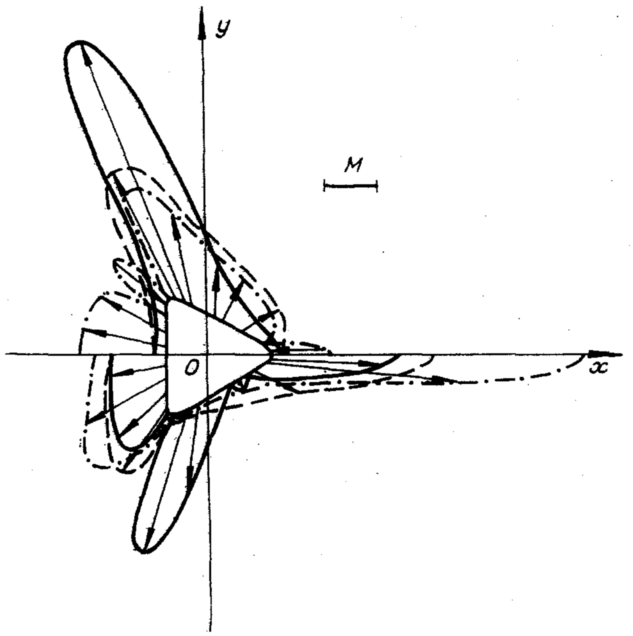

Fig. 1

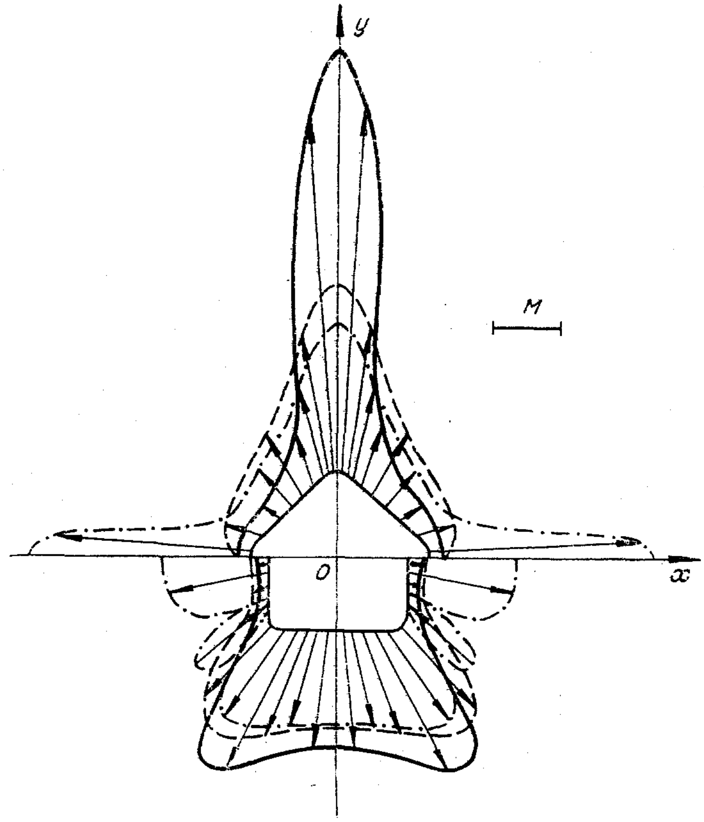

Fig. 2

$$3c_3 m_1 \zeta_1^6 + \zeta_1^4 - m_1 \zeta_1^2 - 3c_3 = 0, \tag{3.1}$$

larger in absolute value than unity ($|\zeta_1^{(k)}| > 1$).

The stress functions are, according to (1.12), of the form

$$\varphi_j'(z_j) = \frac{3a_3^{(j)}\zeta_j^6 + a_1^{(j)}\zeta_j^4 - A_1^{(j)}\zeta_j^2 - 3A_3^{(j)}}{R_j\left[\left(\zeta_j^4 - 3c_3\right) - m_j\left(\zeta_j^2 - 3c_3\zeta_j^6\right)\right]} \quad (j = 1, 2).$$

Numerical values of the bending moments M_θ (in fractions of M) are given in Table 2 at some points of the contour of the square aperture ($c_3 = \pm 1/9$) of the plywood panel with the complex parameters (2.2). The roots of Eq. (3.1) larger in absolute value than unity ($|\zeta_1^{(k)}| > 1$) are correspondingly equal to

$$(\zeta_1^{(k)})^2 = -5.103 - i4.767, \quad \zeta_2^{(k)} = \bar{\zeta}_1^{(k)}, \quad \text{if} \quad E_x = E_{\max},\ c_3 = -\frac{1}{9},$$

$$(\zeta_1^{(k)})^2 = 5.683 + i4.280, \quad \zeta_2^{(k)} = \bar{\zeta}_1^{(k)}, \quad \text{if} \quad E_x = E_{\max},\ c_3 = \frac{1}{9},$$

$$(\zeta_1^{(k)})^2 = 5.103 - i4.767, \quad \zeta_2^{(k)} = \bar{\zeta}_1^{(k)}, \quad \text{if} \quad E_x = E_{\min},\ c_3 = -\frac{1}{9},$$

$$(\zeta_1^{(k)})^2 = -5.683 + i4.280, \quad \zeta_2^{(k)} = \bar{\zeta}_1^{(k)}, \quad \text{if} \quad E_x = E_{\min},\ c_3 = \frac{1}{9},$$

$$(k = 1, 2).$$

The distribution of the moments M_θ along the edge of the corresponding apertures in the plywood panel for which the sides parallel to the Oy axis ($M_x^\infty = M$, $M_y^\infty = 0$) are loaded is illustrated in Figs. 1 and 2. The plots situated in the lower part of Fig. 1 ($\pi \le \theta \le 2\pi$) correspond to the case in which the sides of the plate parallel to the Ox axis ($M_y^\infty = M$, $M_x^\infty = 0$) are loaded. The solid lines correspond to the case $E_x = E_{max}$, the dashed-dot lines correspond to the case $E_x = E_{min}$, and the dashed lines correspond to an isotropic plate with a Poisson coefficient equal to 0.3.

The two-dimensional problem of elasticity theory for an anisotropic plate with an aperture of the form (1.5) is solved analogously.

LITERATURE CITED

1. S. G. Lekhnitskii, Anisotropic Plates [in Russian], GITTL, Moscow (1957).
2. G. N. Savin, Stress Concentration near Apertures [in Russian], GITTL, Moscow—Leningrad (1951).
3. S. G. Lekhnitskii, "An approximate method for the determination of stresses in an elastic anisotropic plate near an aperture differing little from a circular one," Inzh.-Fiz. Zh., 17 (1953).
4. B. I. Ermolaev, "An approximate method for the determination of the stresses upon the deflection of an anisotropic plate with an aperture," Izv. Vyssh. Uchebn. Zaved. Stroit. Arkhit., No. 1 (1960).
5. B. I. Ermolaev, "An approximate solution of the problem of simple deflection of an orthotropic plate with oval and square apertures," Tr. Sarat. Politekh. Inst., 2 (1959).
6. A. S. Kosmodamianskii, "A new approximate method for the determination of the stresses in an anisotropic plate with a curved aperture," in: Some Problems of Elasticity Theory on Stress Concentration and Deformation of Rigid Bodies [in Russian], No. 2, Izd. Saratovsk Univ., Saratov (1965).
7. V. V. Meglinskii, "Some problems of the deflection of thin multiply connected anisotropic plates," in: Some Problems of Elasticity Theory on Stress Concentration and Deformation of Rigid Bodies [in Russian], No. 3, Izd. Saratovsk Univ., Saratov (1967).
8. T. L. Martynovich, V. V. Bozhidarnik, and Yu. M. Maksimovich, "The effect of eccentricity of the reinforcement of the edge of an aperture on the stress state in an anisotropic plate," Mekh. Polim., No. 2 (1974).
9. I. I. Privalov, Introduction to the Theory of Complex Variable Functions [in Russian], GITTL, Moscow (1954).

TEMPERATURE INFLUENCE IN USE OF AN INTERFERENCE METHOD TO STUDY THE EFFECTS OF NORMAL STRESSES

A. A. Sharts

UDC 535.854:531.787

It can now be considered established that deviations in fluid behavior from the classical theory predictions appear mainly in the normal stress effects [1]. For fluids with structural viscosity which varies with the change in the velocity gradient, these effects (Weissenberg effects [2]) are observed at comparatively low velocity gradients and extensive experimental material exists [3].

On the other hand, for fluids which do not exhibit changes in viscosity even for high velocity gradients in both the classical [4] and much later experiments, there are no data on a study of the normal stresses at high velocity gradients in the literature. The absence of such experiments becomes understandable if it is taken into account that while measurement of the viscosity imposes no great demands on the adjustment of the apparatus, a study of the normal stresses requires ultimately accurate surfaces and careful adjustment to reduce the dynamical errors, which is difficult to achieve at those high velocity gradients when the appearance of second-order effects could be expected in fluids with Newtonian viscosity. The influence of nonparallelism in the mounting of disks in a torsion flow was studied in [5].

Small gaps (on the order of tenths of a micron) must be used to achieve high velocity gradients in a torsion flow, and this does not permit application of traditional methods of measuring the normal stresses since both manometer orifices and piezosensors distort the microgeometry of the gap.

A contactless method of investigating the normal stresses in a torsion flow by using the interference of a large path difference and the property of epoxy resin to change the index of refraction with the change in load was proposed in [6].

Since heat is generated in a fluid mass subjected to a shear stress and interference methods are quite sensitive to the thermal shift of optical surfaces, it is necessary to

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Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 177-179, September-October, 1977. Original article submitted September 1, 1976.

0021-8944/77/1805-0735$07.50